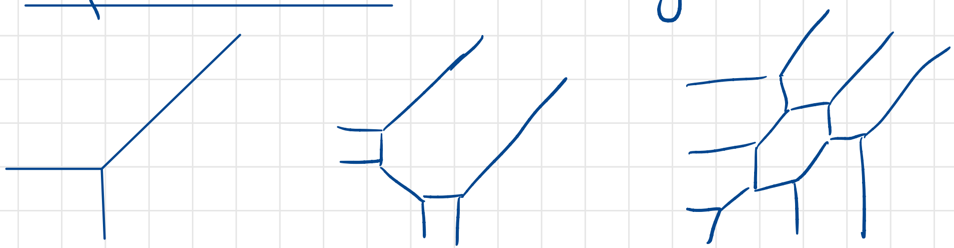
A quadratically enriched tropical Bézout Theorem

jt with Andrés Jaramillo Puentes

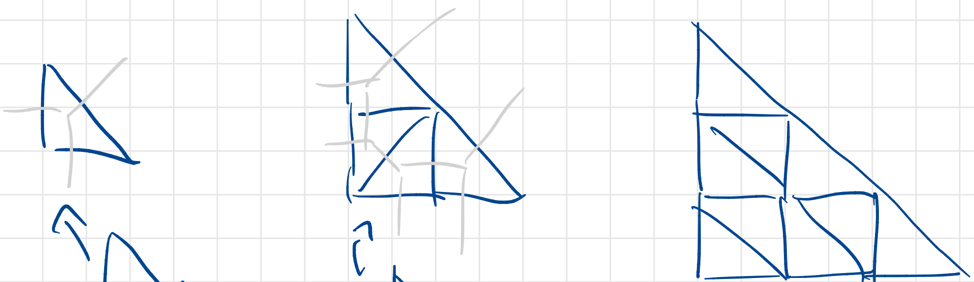
Tropical curves

d = degree

1



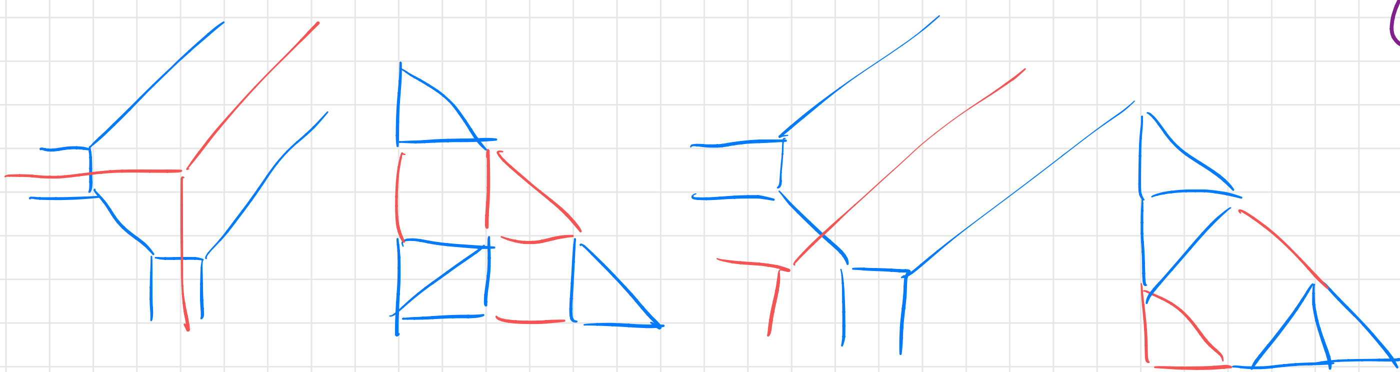
$d=1$ $d=2$ $d=3$



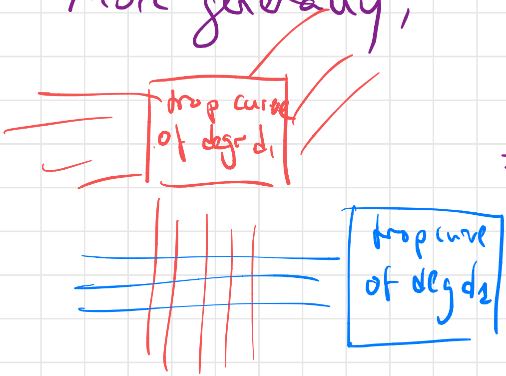
Δ_1 Δ_2

Bézout for tropical curves

Ex:



More generally,

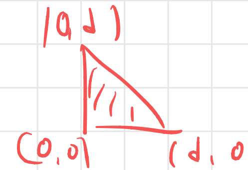


$\Rightarrow$ # intersection pts $= d_1 \cdot d_2$

The combinatorial structure of a tropical curve ~~are~~ is determined by its **dual subdivision**

$\simeq$ subdivision of $\Delta_d = \mathrm{Conv}((0,0),(0,d),(d,0))$

d = degree



Tropical curve	Dual subdivision
vertex	max cells
edges	edges
Conn comp of $\mathbb{R}^2 \setminus$ curve	vertices

st · dual edges are orthogonal
· inclusions are inverted

$C_1 \cup C_2$ dual subdiv of $C_1 \cup C_2$

trop curves

$p \in C_1 \cap C_2 \longleftrightarrow$ Parallelogram

Def $\mathrm{mult}_p(C_1, C_2) :=$ Area of dual parallelogram

Bézout's thm for tropical curves (B. Sturmfels)

C_1, C_2 trop curves of deg d_1 resp d_2

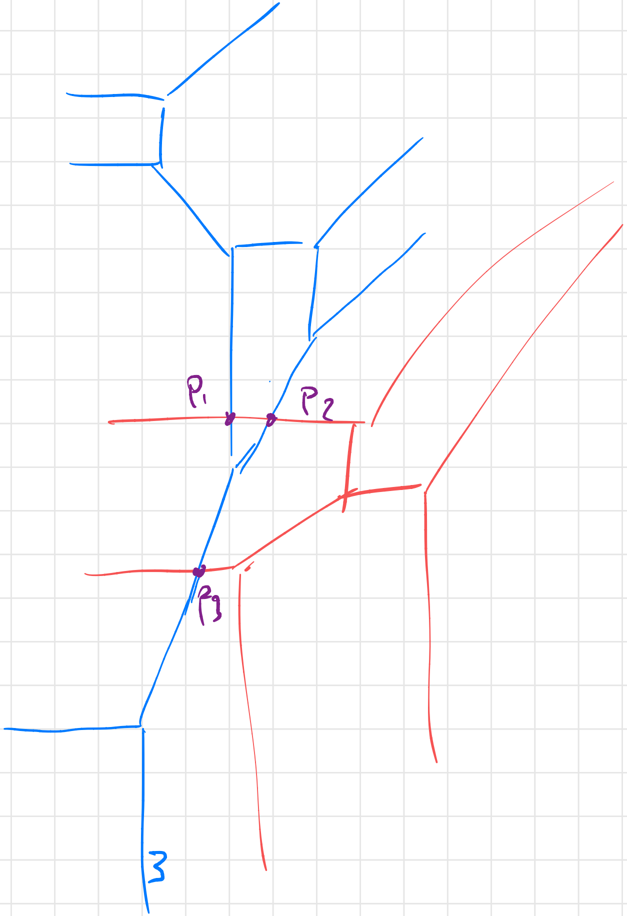
$$\sum_{p \in C_1 \cap C_2} \mathrm{mult}_p(C_1, C_2) = \mathrm{Area}(\Delta_{d_1+d_2}) - \mathrm{Area}(\Delta_{d_1}) - \mathrm{Area}(\Delta_{d_2})$$

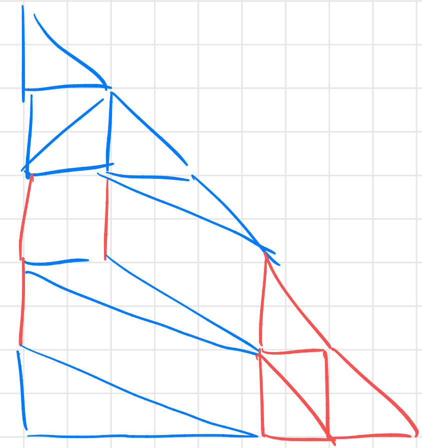
$$= \frac{(d_1+d_2)^2}{2} - \frac{d_1^2}{2} - \frac{d_2^2}{2} = d_1 \cdot d_2$$

(4)

Q: Can we use tropical geometry to solve questions in $\mathbb{A}^1$-enumerative geometry?

ie enriched in $GW(k)$





$$\mathrm{mult}_{p_1}(C_1, C_2) = 1$$

$$\mathrm{mult}_{p_2}(C_1, C_2) = 2$$

$$\mathrm{mult}_{p_3}(C_1, C_2) = 3$$

Bézout for curves enriched in $GW(k)$ (Stephen McKean)

k a field

$C_1 = V(F_1)$ $\qquad$ $C_2 = V(F_2)$ $\qquad \subseteq \mathbb{P}^2_k$

6

F_1: deg d_1 $\qquad$ F_2: deg d_2

Then

$$\sum_{x \in C_1 \cap C_2} \mathrm{Tr}_{k(x)/k} \langle \det \mathrm{Jac}(F_1, F_2)(x) \rangle = \frac{d_1 \cdot d_2}{2} \cdot \mathbb{H} \in GW(k)$$

when $d_1 + d_2$ is odd

rel orientable

Field of Puiseux series

$$k\{\{t\}\} = \left\{ a = \sum_{i=i_0}^{\infty} a_i t^{i/N} \;\middle|\; a_i \in k, \quad N \in \mathbb{N} \right\}$$

Q: What is $GW(k\{\{t\}\})$? $\cong GW(k)$

A: $GW(F)$ is generated by $\langle a \rangle$, $a \in F^\times/(F^\times)^2$

field

Exercise:

$$k\{\{t\}\}^\times / (k\{\{t\}\}^\times)^2 \cong k^\times/(k^\times)^2$$

7

$$\ln : \sum_{i=i_0}^{\infty} a_i t^{i/N} \longmapsto a_{i_0}$$

$(a_{i_0} \neq 0)$

$$F(x,y) = a(t) + b(t)\cdot x + c(t)\cdot y \in k((t))[x,y]$$

deg 1

$$x(t) = x_0\, t^{-v_0} + h.o.t \ldots$$

$$y(t) = y_0\, t^{-w_0} + h.o.t$$

$$a(t) = a_0\, t^{d} + h.o.t$$

$$b(t) = b_0\, t^{e} + h.o.t$$

$$c(t) = c_0\, t^{f} + h.o.t.$$

Want to solve

$$F(x(t), y(t)) = 0$$

$$a_0 \cdot t^{d} + h.o.t.$$

$$+\ b_0\, t^{e-v_0} \cdot x_0 + h.o.t$$

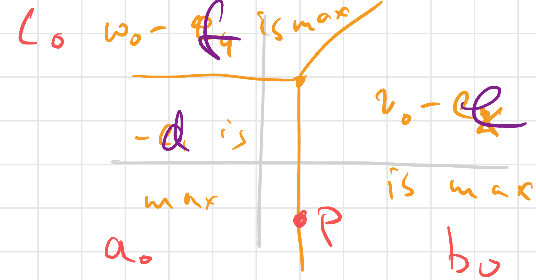
$$+\ c_0\, t^{f-w_0} \cdot y_0 + h.o.t$$

$\rightsquigarrow$ Need ~~min~~ max $(-d, -e+v_0, -f+w_0)$ to be attained at least twice

$d = 1 \qquad v_0 = e - d, = 1$

$e = 2 \qquad w_0 = f - d, = 2$

$f = 3 \qquad w_0 = v_0 + \underbrace{f - e}_{1}$



tropical curve of deg 1

In general $F \in k\{\{t\}\}[x,y]$ of deg d

$\rightsquigarrow$ tropical curve of deg d ① & ②

In fact if $k = \mathbb{C}$, $F_1, F_2 \in \mathbb{C}\{\{t\}\}[x,y]$

$\rightsquigarrow$ 2 tropical curves C_1 & C_2

"definition"

$GW(\mathbb{C}\{\{t\}\}) \cong GW(\mathbb{C}) \cong \mathbb{Z}$

$\text{mult}_p(C_1, C_2) = \#$ of zeros $(x(t), y(t))$ of F_1 and F_2

with $(-v_0, -w_0) = p$

'' initial

$\Rightarrow$ proof of Bézout / $\mathbb{C}\{\{t\}\}$

Q: ~~What about other fields k?~~

What is $\mathrm{Tr}_{E/k\{\{t\}\}}(\langle \det \mathrm{Jac}(F_1, F_2)(x(t), y(t)) \rangle)$?

k arbitrary

coord ring of $(x(t), y(t))$ with $(-v_0, -w_0) = p$

3

Need to remember the initials of the coeff of F_1 & F_2 !

Def An enriched tropical curve $\tilde{C} = (C, (\alpha_I))$ is a tropical curve C together with a coefficient $\alpha_I \in k^\times/(k^\times)^2$ assigned to each comp of $\mathbb{R}^2 \setminus C$ / vertex in dual subdiv

Def Call $(i_1, i_2) = I \in \mathbb{Z}^2$ odd if both i_1 & i_2 are odd.

4

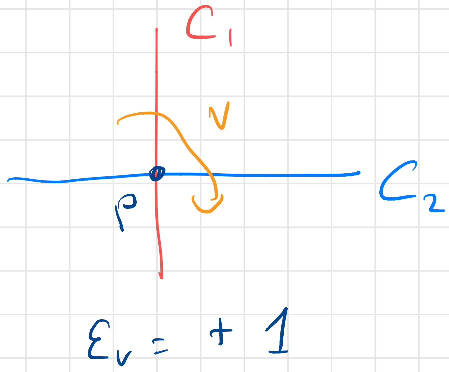
Thm (Jaramillo Puentes - P.)

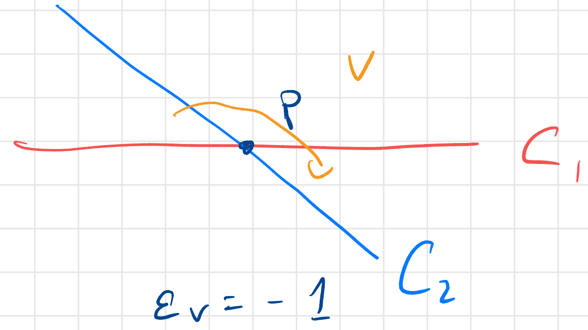
$$\mathrm{Tr}_{E/k((t))} \langle \det \mathrm{Jac}(F_1, F_2)(x(t), y(t)) \rangle$$

$$= \sum_{\substack{v \text{ odd} \\ \text{vertex} \\ \text{of parallelogram} \\ \text{dual to } p}} \langle \varepsilon_v \alpha_v \rangle + \text{hyperbolic forms}$$

α_v = coeff of v

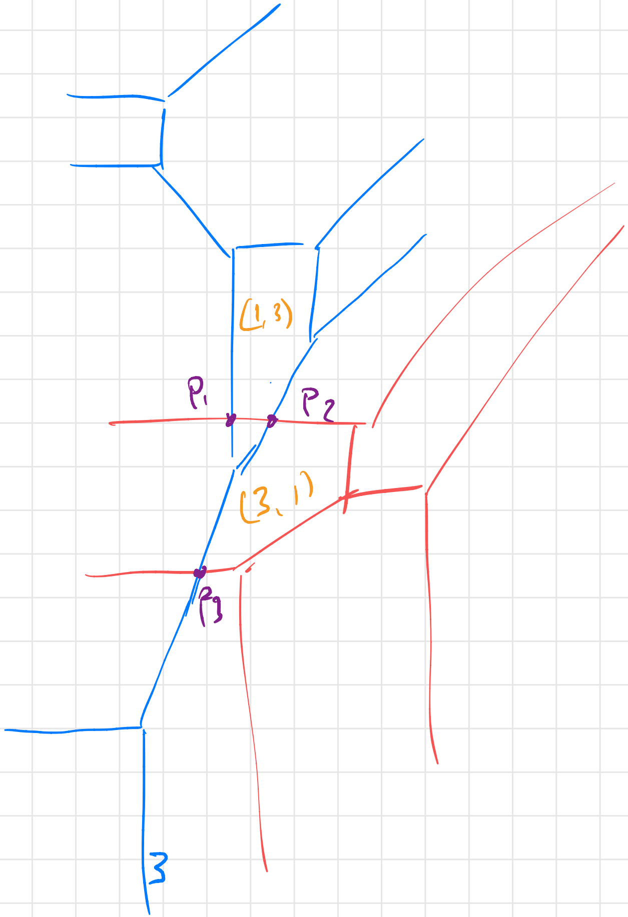
ε_v = sign

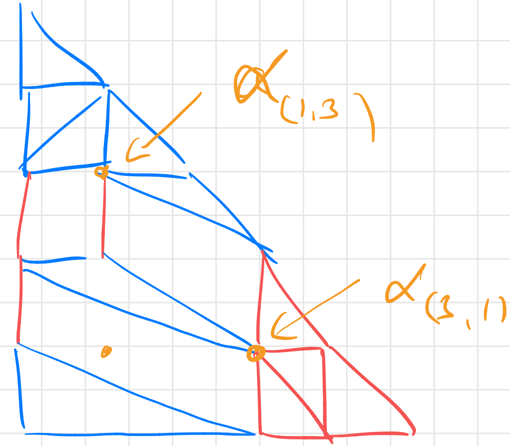




<u>Cor</u>: Bézout in rel orientable case !

$d_1 + d_2$ odd



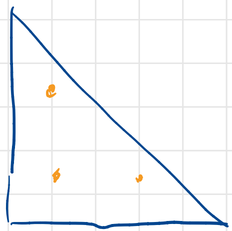


$$\widetilde{mult}_{p_1}(\tilde{C}_1, \tilde{C}_2) = \langle -\alpha_{(1,3)} \rangle$$

$$\widetilde{mult}_{p_2}(\tilde{C}_1, \tilde{C}_2) = \langle \alpha_{(1,3)} \rangle + \langle \alpha_{(3,1)} \rangle$$

$$\widetilde{mult}_{p_3}(\tilde{C}_1, \tilde{C}_3) = \langle -\alpha_{(3,1)} \rangle + h$$

Why $d_1 + d_2$ odd?



$d_1 + d_2$ odd $\Rightarrow$ odd lattice points lie in the interior

$\Delta_{d_1+d_2}$

Generalizations: · can define "enriched tropical hypersurfaces" (higher dimensions)

$\rightsquigarrow$ Bézout not only for curves

· can compute intersections in toric varieties

$\rightsquigarrow$ enriched Bernstein Kushnirenko

· can also say sth about non-orientable ones